(*) Quadratics. (S6.1 – 6.3)

- Quadratics are best studied in the form of

$$f(x) = a(x-b)^2 + c = ax^2 - 2abx + (ab^2 + c), \quad a \neq 0.$$

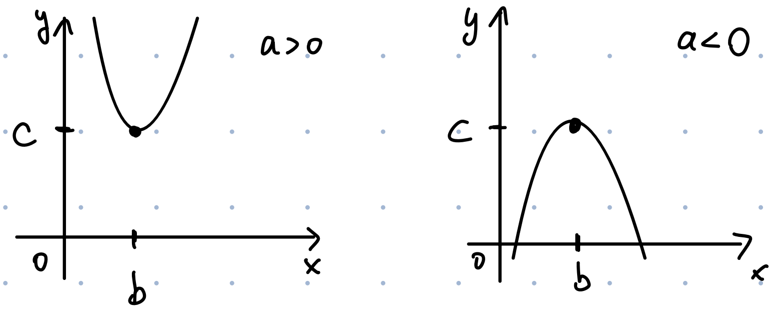
- Stationary pt: $f'(x) = 2ax - 2ab = 2a(x-b) = 0$.

$\Rightarrow$ $x = b$ is the stationary point, $f(b) = c$.

Second derivative test: $f''(x) = 2a$

$\Rightarrow \begin{pmatrix} x=b \\ y=c \end{pmatrix}$ is $\begin{cases} \text{a local min if } a>0 & \leftarrow \text{vertex.} \\ \text{a local max if } a<0. \end{cases}$

- Graph of $f(x) = a(x-b)^2 + c$ :




- One can determine a quadratic function with <u>three points</u> on it.
  (or <u>vertex + one point</u>).

<u>Ex</u>. Find the quadratic that goes through vertex $(-1,-4)$ and $(1,0)$.

<u>Sol.^n</u> Assume the quadratic function is

$$f(x) = a(x-b)^2 + c.$$

- Vertex: $(b,c) = (-1,-4)$, $f(x) = a(x+1)^2 - 4$.
- $x = 1$, $f(1) = 4a - 4 = 0 \Rightarrow a = 1$.

So $f(x) = (x+2)^2 - 4$.

- If we know <u>two zeros ($x_1, x_2$) and one point</u>, we can very easily know the quadratics: $f(x)$ must be $a(x-x_1)(x-x_2)$.

<u>Ex</u>. Find the quadratic that goes through $(-1,0)$, $(3,0)$ and $(0,-6)$.

<u>Sol.^n</u> $f(x) = a(x+1)(x-3)$.

- $x=0$, $f(0) = a(1)(-3) = -6$, so $a = 2$.

So $f(x) = 2(x+1)(x-3)$.

- A quadratic may have <u>0, 1 or 2 zeros</u>: for $f(x) = Ax^2 + Bx + C$,
- Determinant $= B^2 - 4AC$.
- Determinant $> 0$ : <u>two</u> distinct zeros.
- Determinant $= 0$ : <u>one</u> zero.
- Determinant $< 0$ : No zeros.

<u>Ex</u>. Explain why $f(x) = 100x^2 + 5x + 1$ is always positive.

<u>Sol.</u>[n] Determinant $= 5^2 - 4(100)(1) = 25 - 400 = -375 < 0$.
So $f(x)$ has no zero $\Rightarrow$ $f(x)$ is either always positive or always negative.
But $f(0) = 1 > 0$, so $f(x)$ is always positive.

## ⊛ <u>Cubics</u> (§11.1)

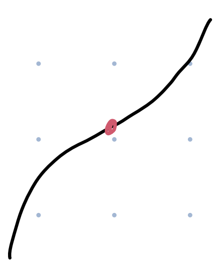
The cubics are of the form:

$$f(x) = Ax^3 + Bx^2 + Cx + D. \qquad A \neq 0.$$

- Stationary points : $f'(x) = 3Ax^2 + 2Bx + C = 0$.
- $f'$ may have <u>0, 1, or 2 zeros</u>.

① $f'$ has no zero :

- A cubic always has one point of inflection on $(-\infty, \infty)$.



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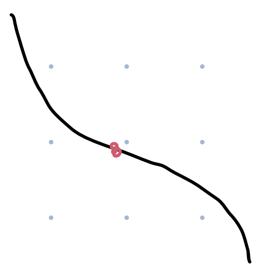
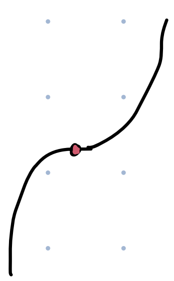


② $f'$ has one zero :

- the stationary point is also a point of inflection.



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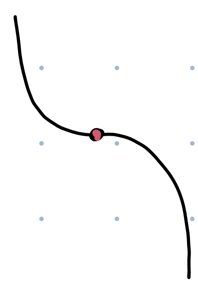
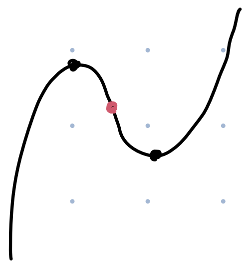


③ $f'$ has two zeros:



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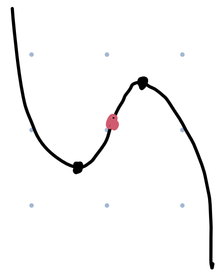
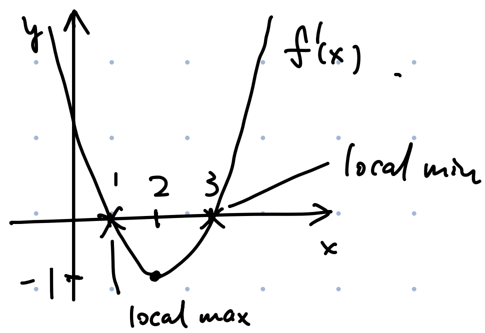


• there is a point of inflection, as the midpoint of two stationary points.

**Ex.** Plot $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 2$.

**Sol.$^n$** ① Stationary points: $f'(x) = x^2 - 4x + 3 = (x-1)(x-3) = 0$.

$x = 1$ & $x = 3$.




$f'(x) = x^2 - 4x + 4 - 1 = (x-2)^2 - 1$

• $x = 1$, $f(1) = \frac{1}{3} - 2 + 3 + 2 = \frac{10}{3}$. ← local max

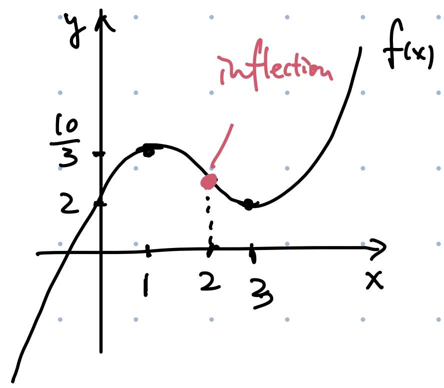
• $x = 3$, $f(3) = \frac{1}{3}(3)^3 - 2(3)^2 + 9 + 2$ ← local min

$= 9 - 18 + 9 + 2 = 2$

② Inflection points:

$f''(x) = 2x - 4 = 0$ when $x = 2$.

$f(2) = \frac{1}{3}2^3 - 2(2)^2 + 6 + 2 = \frac{8}{3} \cancel{-8} \cancel{+8} = \frac{8}{3}$




• It is not guaranteed one can always write down the zeros of a quadratic function: one often has to guess a zero, and then factorise.

**Ex.** Find the zeros of $f(x) = x^3 - 21x - 20$. Plot $f(x)$.

**Sol.$^n$** $\boxed{\text{I}}$ $f(\boxed{-1}) \overset{①}{=} (-1)^3 - 21(-1) - 20 = 0$ So $(x-(-1)) = (x+1)$ must be a factor.

Let $x^3 - 21x - 20 = (x+1)(ax^2 + bx + c) = ax^3 + (a+b)x^2 + (b+c)x + c$

$= x^3 + 0 - 21x - 20$

$\Rightarrow a = 1,\ b = -1,\ c = -20$. So $f(x) = (x+1)(x^2 - x - 20)$

Let $x^2 - x - 20 = 0$, $x^2 - x + \frac{1}{4} = (x - \frac{1}{2})^2 = \frac{81}{4}$, $x = \frac{\pm 9}{2} + \frac{1}{2} = \overset{②}{5}$ or $\overset{③}{-4}$

• Three roots of $f$: $x = -1$, $x = 5$, $x = -4$.

$\boxed{\text{II}}$ • Stationary points: $f'(x) = 3x^2 - 21 = 0$, $x = \pm\sqrt{7}$.

$f''(x) = 6x$

• $x = \sqrt{7} \overset{2.64}{\approx}$, $f''(x) = 6\sqrt{7} > 0 \Rightarrow$ local min

• $x = -\sqrt{7}$, $f''(x) = -6\sqrt{7} < 0 \Rightarrow$ local max.

• Inflection pt: $f''(x) = 6x = 0$, $x = 0$.

• Infinity behaviour: $f(x) = x^3 - 21x - 20$,

$$\begin{cases} \text{As } x \to \infty,\ x = \text{large},\ f(x) = \text{large}^3 + \cdots = \text{large},\ \lim_{x\to\infty} f(x) = \infty. \\ \text{As } x \to -\infty,\ x = -\text{large},\ f(x) = (-\text{large})^3 + \cdots = -\text{large},\ \lim_{x\to\infty} f(x) = -\infty. \end{cases}$$