⊛ Polynomials (§ 11.2, 11.3)

Def.^{n} (Polynomials) f is a <u>polynomial</u> function of <u>degree</u> n, if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

for constants $a_0, \cdots, a_n$, and $a_n \neq 0$.

- Terms like $\sqrt{x}$ or $\frac{1}{x}$ cannot appear in a polynomial.

<u>Goal</u>: make rough plots of polynomial functions.

- Zeros of a polynomial:
  - A polynomial of degree n may have up to n zeros.
  - If a polynomial f of degree n has exactly n distinct zeros $c_1, \cdots, c_n$, then there is some number $a \neq 0$ that

$$f(x) = a(x-c_1)(x-c_2)(x-c_3)\cdots(x-c_n).$$

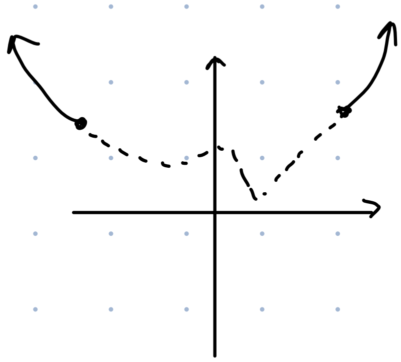
<u>Ex</u>. Find the fourth degree polynomial that goes through $(1,0)$, $(2,0)$, $(3,0)$, $(4,0)$ and $(0,12)$.

<u>Sol.^{n}</u>
- Zero $(1,0)$, $(2,0)$, $(3,0)$, $(4,0)$.
- $f(x) = a(x-1)(x-2)(x-3)(x-4)$.
- $x=0$, $f(0) = a(-1)(-2)(-3)(-4) = 24a = 12$, $a = \frac{1}{2}$,
- So $f(x) = \frac{1}{2}(x-1)(x-2)(x-3)(x-4)$.

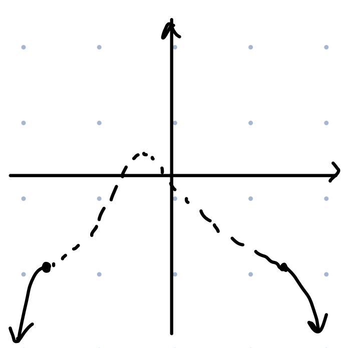
$\boxed{\Sigma}$

- Infinity behaviour of even/odd degree polynomials:
- $f = a_0 + a_1 x + \cdots + a_n x^n$
- n = even:

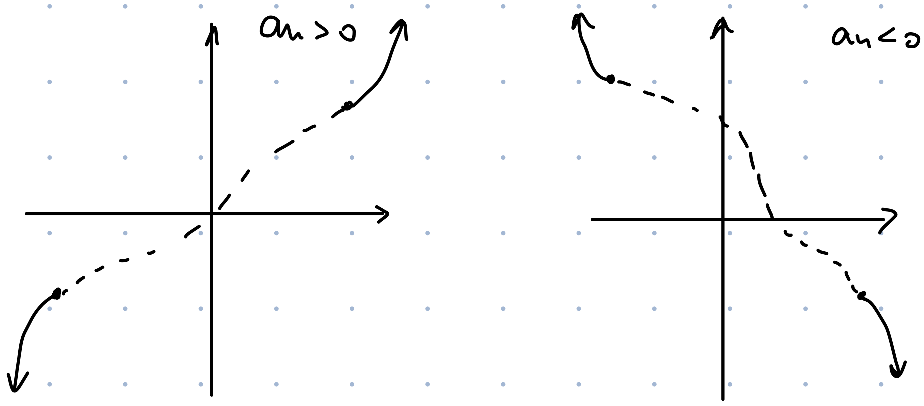
$a_n > 0$



$a_n < 0$



- n = odd :



- If $f(x)$ is even, then $f(x)$ cannot have x^{odd} terms.
- If $f(x)$ is odd, then $f(x)$ cannot have x^{even} terms.

Ex. $x^4 + 1$ is even, $5x^3 - 10x$ is odd, $x+1$ is neither even nor odd.

- Derivative of deg n polynomial is a deg $(n-1)$ polynomial. (For $n \geq 1$).
- A deg n polynomial may have up to $(n-1)$ stationary points.

End of midterm 2

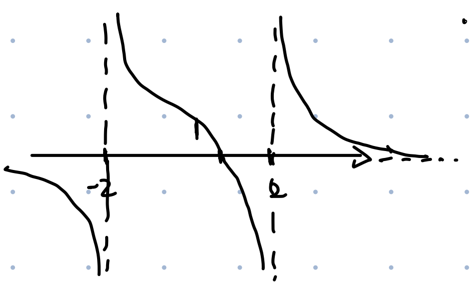
⊛ Rational functions (S11.4).

- Rational functions : functions of form $\frac{\text{polynomial}}{\text{polynomial}}$.
- Let $f(x) = \frac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials.

Ex. For $f(x) = \frac{x-1}{x^2-4}$, find all asymptotes.

Sol^n $f(x) = \frac{x-1}{(x-2)(x+2)}$, denominator = 0 when $x=2$, $x=-2$.

- Vertical asymptote at $x=2$ and $x=-2$.
- As $x \to \infty$, $\lim_{x\to\infty} \frac{x-1}{x^2-4} = \lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\text{large}} = \text{small} = 0$.
- As $x \to -\infty$, $\lim_{x\to-\infty} \frac{x-1}{x^2-4} = \lim_{x\to-\infty} \frac{x}{x^2} = \lim_{x\to-\infty} \frac{1}{x} = \frac{1}{-\text{large}} = -\text{small} = 0$.
- So horizontal asympotote at $y=0$.



- When $x > 2$, $\frac{\overset{+}{x-1}}{\underset{+}{(x-2)}\underset{+}{(x+2)}} > 0$.
- $1 < x < 2$: $\frac{+}{-\cdot+} < 0$, $-2 < x < 1$: $\frac{-}{-\cdot+} > 0$
- $x < -2$: $\frac{-}{-\cdot-} < 0$.

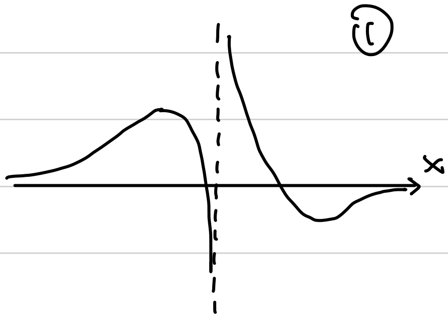
- Rational function can only change sign at its zeros.

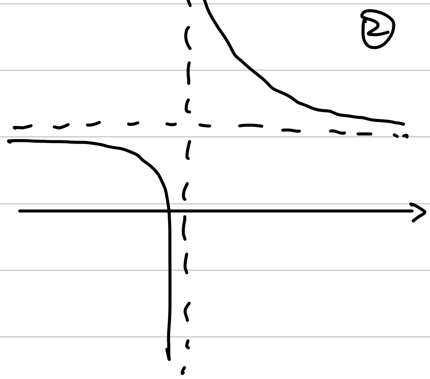
- Horizontal asympotoes:

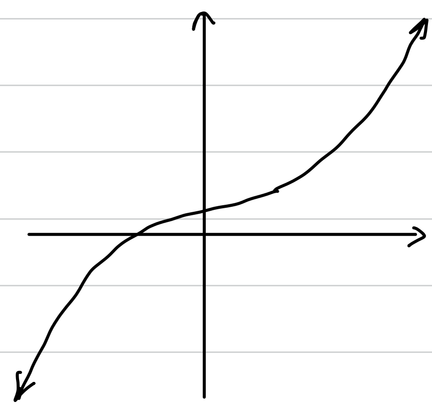
  ① $\frac{\text{small deg}}{\text{large deg}}$: horizontal asymptote at $y=0$.

  ② $\frac{\text{equal deg}}{\text{equal deg}}$: horizontal asymptote at $y = \frac{\text{leading coeff}}{\text{leading coeff}}$.

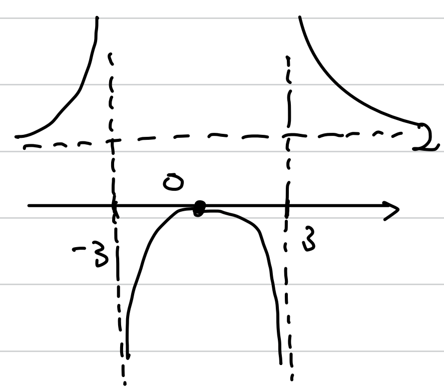
  ③ $\frac{\text{large deg}}{\text{small deg}}$: $y \to \infty$ or $-\infty$ depending on different scenarios.







Ex. Find all asymptotes of $f(x) = \frac{2(x^3-3x^2)}{(x+3)(x^2-6x+9)}$



Sol.ⁿ • First simplify the terms: $2(x^3-3x^2) = 2x^2(x-3)$,

- $(x+3)(x^2-6x+9) = (x+3)(x-3)^2$,

So $f(x) = \frac{2x^2(x-3)}{(x+3)(x-3)^2} = \frac{2x^2}{(x+3)(x-3)} = \frac{2x^2}{x^2-9}$ even even

- Vertical asymptote: $x=3$, $x=-3$
- Zero: $x=0$.
- As $x\to\infty$, $\lim_{x\to\infty}\frac{2x^2}{(x+3)(x-3)} = \lim_{x\to\infty}\frac{2x^2}{x^2-9} = \lim_{x\to\infty}\frac{2x^2}{x^2} = \lim_{x\to\infty} 2 = 2.$ leading leading

- Horizontal asymptote: $y=2$.

- $x\to\infty$, $\frac{2x^2}{x^2-9} \geq \frac{2x^2}{x^2} = 2$
- x in $(0,3)$: $\frac{2x^2}{(x+3)(x-3)} = \frac{+}{+\ -} < 0$

$\boxed{h}$

Ex. Find the global/absolute value of $f(x) = \dfrac{x-3}{x^2+7}$ on $(-\infty, \infty)$.

Sol^{n}: • x^2+4 is always positive, so no vertical asymptote.

$\boxed{1}$ Critical pt: $f'(x) = \dfrac{\text{num}' \cdot \text{denom} - \text{num} \cdot \text{denom}'}{\text{denom}^2}$ (quotient rule.)

x^2-6x-7
$= (x-$

$$= \frac{x^2+7-(x-3)\,2x}{(x^2+7)^2} = \frac{x^2+7-2x^2+6x}{(x^2+7)^2} = \frac{-x^2+6x+7}{(x^2+7)^2} = 0,$$

when $-x^2+6x+7 = -(x^2-6x-7) = -(x-7)(x+1) = 0$,

that is $x = 7$ ①, $x = -1$ ②.

• nowhere $f(x)$ is not differentiable.

• infinities: $x = -\infty$ ③, $x = +\infty$ ④.

$\boxed{2}$ Evaluation: ① $f(7) = \dfrac{7-3}{49+7} = \dfrac{4}{56} = \dfrac{1}{14}$ ← global max

② $f(-1) = \dfrac{-1-3}{1+7} = -\dfrac{1}{2}$ ← global min.

③, ④ deg of numerator > deg of denominator, so horizontal asymptote: $y = 0$.