Ⓕ Exponential functions (S9.1, 9.2, 9.3)

<u>Motivation</u>: in microbiology, the speed of the population of microbe grows is often proportional to the population itself.

- Polynomials don't do the job we need. We need more functions.

<u>Ex</u>. Let $f(t)$ = count of bacterias in a dish at t hours. We know
- the bacteria count doubles every hour.
- $f(0) = 5$.

Find the function $f(t)$.

<u>Sol.^{n}</u>

t	0	1	2	3	4	5
$f(t)$	5	10	20	40	80	160
f will increase	5	10	20	40	80	/

$\Delta f = f(t+1) - f(t)$

$f(1) = 5 \times 2$

$f(2) = 5 \times 2 \times 2 = 5 \times 2^2$

$f(3) = 5 \times 2 \times 2 \times 2 = 5 \times 2^3$.

...

$f(t) = 5 \times \overbrace{2 \times \cdots \times 2}^{t \text{ times}} = 5 \times 2^t$.

- At each time t, the amount of f will increase in the next hour is proportional to $f(t)$.

<u>Def.^{n}</u> <u>Exponential functions</u> are $f(x) = C b^x$, where $C \neq 0$, $b > 0$ are constants

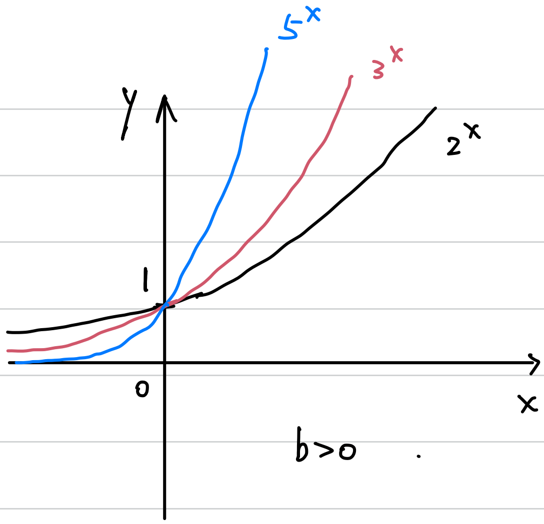
Two prototypes:

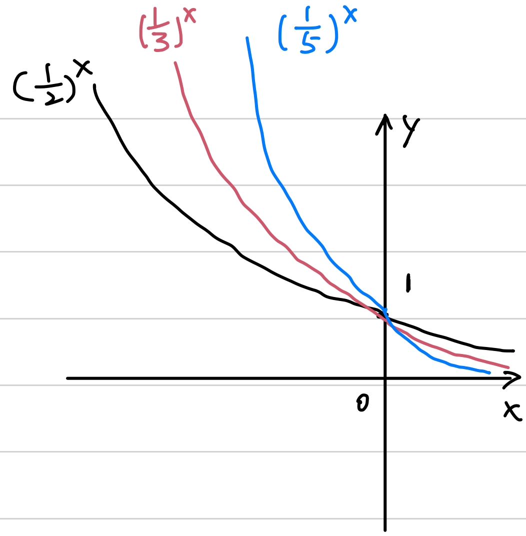
- when $b > 1$, function b^x <u>increases</u>, and b^x increases at a rate proportional to itself. We call b^x <u>exponentially increasing (growing)</u>:

→ Larger b is, the faster b^x increases.

→ $b^x < 1$ for $x < 0$, and $b^x > 1$ for $x > 0$.

→ $\lim_{x \to -\infty} b^x = 0$.





- When $0<b<1$, function b^x <u>decreases</u> at a rate proportional to itself. We call b^x <u>exponentially decreasing</u> (decaying).
  - → Smaller b is, faster b^x decreases.
  - → $b^x > 1$ on $x<0$, $b^x<1$ on $x>0$.
  - → $\lim_{x\to\infty} b^x = 0$.
- In both $b>1$ and $0<b<1$,
  - → $b^x = 1$ at $x=0$.
  - → $b^x > 0$ at any x.

- Quick review of exponentials: for $b>0$,
  1. $b^{x_1} b^{x_2} = b^{x_1+x_2}$
  2. $\dfrac{b^{x_1}}{b^{x_2}} = b^{x_1-x_2}$
  3. $(b^{x_1})^{x_2} = b^{x_1 x_2}$.

<u>Ex</u>. A saving account promotes a monthly rate of 1%. I deposited 100 \$ into the account at month 0. Find the formula for $f(t)$ = balance at month t. Compute the yearly rate of return.

<u>Sol.^{n}</u>. Let $f(t) = Cb^t$, where $b>1$.

- $t=0$, $f(0) = Cb^0 = C = 100$, $\quad f(t) = 100\, b^t$.
- $t=1$, $f(1) = Cb^1 = 100b = 100 \times (1+1\%) = 101$, so $b = 1.01$
- $f(t) = 100(1.01)^t$.
- $f(12) = 100(1.01)^{12} \approx 100(1.1268) = 112.68$.

• APY = yearly return = $\frac{112.68 - 100}{100} = 12.68\%$.

Ex. The mass of radioactive substances = $f(t)$ (kg), at t (months). It is known:

• The mass decays exponentially in time.

• After 2 month, there is 160 kg left.

• After 6 month, there is 10 kg left.

Use those information, find out how much substances are there at the beginning, and how much is left after 10 months. Find the half-life of the substance.

Sol.^{n}. Assume $f(t) = Cb^t$. Since $f(t)$ is exponentially decreasing, $0<b<1$.

$$\begin{cases} f(2) = Cb^2 = 160 & (1) \\ f(6) = Cb^6 = 10 & (2) \end{cases}$$

(1) divided by (2): $\frac{Cb^2}{Cb^6} = \frac{160}{10}$, $\frac{1}{b^4} = 16$, $b^4 = \frac{1}{16}$, $b^2 = \frac{1}{4}$, $b = \frac{1}{2}$.

• $f(2) = C(\frac{1}{2})^2 = 160$, $C = 160 \times 4 = 640$.

So $f(t) = 640\,(\frac{1}{2})^t$.

• At beginning: $f(0) = 640(\frac{1}{2})^0 = 640$ (kg)

• At 10 month: $f(10) = 640\,(\frac{1}{2})^{10} = \frac{640}{1024} = 0.625$ (kg).

• Half-life: amount of time for the mass to decay to its half.

• That is, find t that $f(t) = \frac{1}{2}f(0) = \frac{1}{2}(640) = 320$,

let $f(t) = 640\,(\frac{1}{2})^t = 320$. So $(\frac{1}{2})^t = \frac{1}{2}$, $t = 1$.

$\Rightarrow$ Half-life = 1 month.

Ex. A microbe has population 20 at t days. We know it grows exponentially and it takes 9 days for the microbe to double its population. How large is the population at 30 days in?

Sol.^{n}. let $f(t) = Cb^t$. $f(t)$ grows exponentially, so $b>1$.

· $f(0) = Cb^0 = C = 20$. $\quad f(t) = 20b^t$.

· $f(9) = 20b^9 = 20 \times 2 = 40$, $\quad$ so $b^9 = 2$, $b = (b^9)^{\frac{1}{9}} = 2^{\frac{1}{9}}$.

· $f(t) = 20(2^{\frac{1}{9}})^t = 20(2)^{t/9}$.

· $f(30) = 20(2)^{\frac{30}{9}} = 20(2)^{\frac{10}{3}} \approx 20 \times 10.1 = 202$.