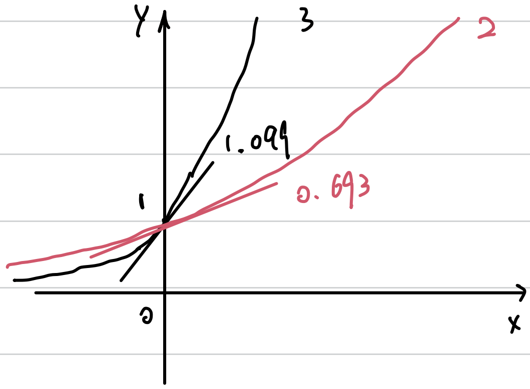
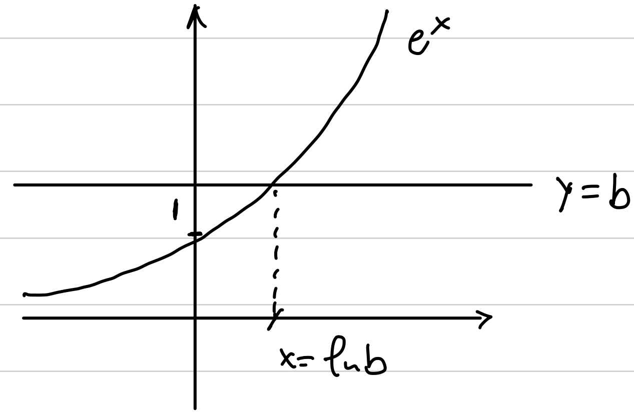
(*) Derivative of exponential functions (§ 9.4).

- For $f(x) = b^x$, $f'(x) = a f(x) = a b^x$, where $a = f'(0)$.
- This is NOT practical in computing derivatives of exponentials.





Q. Is there some b where $f'(x) = f(x)$, that is $f'(0) = 1$?

Def^n: We define e to be the positive number that the slope of tangent line to e^x at $x=0$ is exactly 1. Then the natural exponential function e^x has:

$$\frac{d}{dx}(e^x) = e^x.$$

- $e \approx 2.718$. We call e the Euler number.

Thm Let $k \neq 0$ be a real number. Then

$$\frac{d}{dx} e^{kx} = k e^{kx}.$$

Ex. Compute $\frac{d}{dx}\left(x^2 e^{2x}\right)$.

Sol^n: $(x^2 e^{2x})' = 2x e^{2x} + x^2\, 2e^{2x}$ (Chain rule.)

$= 2(x^2 + x) e^{2x}$.

Ex: Compute $\frac{d}{dx}\left(\frac{e^{2x}}{x+1}\right)\Big|_{x=0}$

Sol^n: $\frac{d}{dx}\left(\frac{e^{2x}}{x+1}\right) = \frac{2e^{2x}(x+1) - e^{2x}}{(x+1)^2} = \frac{e^{2x}(2x+1)}{(x+1)^2}$.

$$\frac{d}{dx}\left(\frac{e^{2x}}{x+1}\right)\Big|_{x=0} = \frac{e^{2(0)}(0+1)}{(0+1)^2} = \frac{1}{1} = 1.$$

☐

• Quick preview of natural logarithms:

Def^n. Let $b>0$. Then we define the <u>natural log of b</u>, $\ln(b) = a$, that $b = e^a$.

That is $\ln(b)$ is the number that $e^{\ln b} = b$.

Ex. • $\ln(1) = 0$: $e^0 = 1$.

• $\ln(e) = 1$: $e^1 = e$.

• $\ln(e^2) = 2$: $e^2 = e^2$.

• $\ln(e^{-2}) = -2$: $e^{-2} = e^{-2}$.

• Derivative of exponential functions: let $f(x) = Cb^x$.

Thm: $\begin{cases} (Cb^x)' = C(\ln b)\, b^x. \\ f'(0) = C\ln(b). \end{cases}$

Pf: $f(x) = Cb^x = C(e^{\ln b})^x$ (since $b = e^{\ln b}$)

$= C e^{(\ln b)x}$ (since $(e^r)^x = e^{rx}$)

Now $f'(x) = C(e^{(\ln b)x})'$

$= C(\ln b)\, e^{(\ln b)x}$ (by $(e^{kx})' = ke^{kx}$, $k = \ln b$.)

$= C(\ln b)\, b^x$, as desired.

And $f'(0) = C(\ln b) b^0 = C\ln(b)$.

Ex. Compute derivative of $f(x) = 10(2)^x$.

Sol.^{n} • $f(x) = 10(e^{\ln 2})^x = 10e^{(\ln 2)x}$

• $f'(x) = (10e^{(\ln 2)x})' = 10(\ln 2)\, e^{(\ln 2)x} = 10(\ln 2)\, 2^x$.

Ex. Compute derivative of $f(x) = -7\left(\frac{1}{3}\right)^x$.

Sol.^{n} • $f(x) = -7\left(e^{\ln(\frac{1}{3})}\right)^x = -7e^{(\ln\frac{1}{3})x}$

• $f'(x) = -7\left(\ln\frac{1}{3}\right) e^{(\ln 3)x} = -7\ln\left(\frac{1}{3}\right)(3)^x$.

Ex. Find the stationary point of $f(x) = e^x + e^{-x}$. Is it a local max or min?

Sol.ⁿ $f'(x) = 1e^x + (-1)e^{-x} = e^x - e^{-x} = 0$,

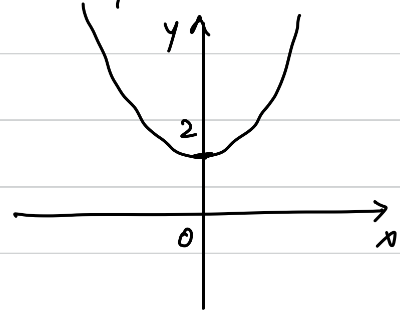
$e^x - e^{-x} = 0$, $e^{2x} - 1 = 0$, $e^{2x} = 1$, but this only happens at $2x = 0$,

that is $x = 0$.

$f''(x) = e^x - (-1)e^{-x} = e^x + e^{-x}$.

$f''(0) = e^0 + e^{-0} = 2 > 0$.

· Second derivative test: a local min at $x = 0$.



Ex. The population of microbes is 100 at time 0 (day), and the instantaneous rate of increase at time 0 (day) is 200. Find the population at 30 days.

Sol.ⁿ Let $f(t) = Cb^t$,

- $f(0) = Cb^0 = C = 100$, $f(t) = 100b^t$.
- $f'(t) = 100(b^t)' = 100(\ln b)b^t$.
- $f'(0) = 100(\ln b)b^0 = 100\ln b = 200$. So $\ln b = 2$, $b = e^{\ln b} = e^2$.
- $f(t) = 100(e^2)^t = 100e^{2t}$.
- $f(30) = 100e^{2(30)} = 100e^{60}$. ← population at 30 days.