⊛ Logarithms (§13.1, 13.2, 13.3, 13.4)

- <u>Motivation</u>: what is the x such that $2^x = 10$?
- Let $f(x) = 2^x$ be an exponential function, then for $f(x) = 2^x = 10$, all we want is $x = f^{-1}(10)$, where f^{-1} is the inverse function to an exponential function.

<u>Defn</u>. Let $b>0$, $b \neq 1$. The logarithm function $g(x) = \log_b(x)$ defined for $x>0$, is the inverse function of $f(x) = b^x$.

- That is, $\log_b(x) = y$ is the unique number y that $b^y = x$.
- When $b=e$, the <u>natural logarithm</u> $\ln(x) = \log_e(x)$.
- When $b=10$, we sometimes write $\log(x) = \log_{10}(x)$.

<u>Ex</u>. ① $\log_2 2 = 1$ since $2^1 = 2$.
② $\log_2 8 = 3$ since $2^3 = 8$.
③ $\ln e = \log_e e = 1$, since $e^1 = e$.
④ $\ln(e^2) = \log_e e^2 = 2$, since $e^2 = e^2$.
⑤ $\log_{10} 1 = 0$ since $10^0 = 1$.
⑥ $\log_{10} 0.01 = -2$ since $10^{-2} = 0.01$.
⑦ $\log_{\frac{1}{2}} 8 = -3$ since $(\frac{1}{2})^{-3} = 2^3 = 8$.

- When $b>1$, if $x>y$, then $\log_b x > \log_b y$.

<u>Ex</u>. Estimate $\log_2 60$.

<u>Soln</u>
- $2^5 = 32 < 60$, so $5 = \log_2 32 < \log_2 60$.
- $2^6 = 64 > 60$, so $\log_2 60 < \log_2 64 = 6$.
- So $5 < \log_2 60 < 6$ and it's closer to 6. ($\log_2 60 \approx 5.9$).

- Algebraic rules for logarithms:

Let $b>0$ and $b \neq 1$. Then:
- $\log_b b = 1$, ($b^1 = b$)
- $\log_b 1 = 0$. ($b^0 = 1$)

Thm Let $b>0$ and $b\neq 1$ Then for any $x,y>0$, any k real:

① $\log_b(xy) = \log_b(x) + \log_b(y)$.

② $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$.

③ $\log_b(x^k) = k\log_b(x)$.

Pf. ① Note $e^{\log_b(x)} = x$, $e^{\log_b(y)} = y$,

then $e^{\log_b(x)+\log_b(y)} = e^{\log_b(x)} \cdot e^{\log_b(y)} = x\cdot y$.

this implies $\log_b(xy) = \log_b(x) + \log_b(y)$.

② Similar to ①.

③ Note $b^{\log_b(x)} = x$. Then

$$b^{k\log_b(x)} = (b^{\log_b(x)})^k = x^k.$$

So $\log_b(x^k) = k\log_b(x)$. $\boxtimes$

Ex. ① $\log_2(16) = \log_2(2^4) = 4\log_2(2) = 4(1) = 4$.

② $5^{(\log_5(x^4)+\log_5(x^2))} = 5^{\log_5(x^4)}\, 5^{\log_5(x^2)} = x^4\cdot x^2 = x^6$.

③ $e^{-\ln(\sqrt{x})} = \frac{1}{e^{\ln\sqrt{x}}} = \frac{1}{\sqrt{x}}$.

④ $\ln(x^3-1) - \ln(x-1) = \ln\left(\frac{x^3-1}{x-1}\right) = \ln\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$

$= \ln(x^2+x+1)$.

• Change of base: logarithm can be written using logarithm with other bases

Thm. Let b,k be positive but not 1. Then

$$\log_b x = \frac{\log_k x}{\log_k b}.$$

• A most frequently used form: $k=e$: $\log_b(x) = \frac{\ln x}{\ln b}$.

Pf: Let $\log_b x = c$. Then $b^c = x$. Take $\log_k$ over each side:

$\log_k b^c = \log_k x$,

$c\log_k b = \log_k x$,

$c = \frac{\log_k x}{\log_b x}$ as desired. $\boxtimes$

Ex Solve $2^{4x-1} = 31$ using $\ln$.

Sol.^{n} Take $\ln$ of both sides:

$$\ln(2^{4x-1}) = \ln(31)$$
$$(4x-1)\ln 2 = \ln 31$$
$$4x = \frac{\ln 31}{\ln 2} + 1$$
$$x = \frac{\ln 31}{4\ln 2} + \frac{1}{4}.$$

Ex. Solve $4^{x+1} - \frac{50}{4^x} = 0$ using $\ln$.

Sol.^{n} Take $\ln$ of both sides

$$\ln(4^{x+1}) = \ln\left(\frac{50}{4^x}\right)$$
$$(x+1)\ln 4 = \ln(50) - \ln(4^x)$$
$$(x+1)\ln 4 = \ln(50) - x\ln 4$$
$$(2x+1)\ln 4 = \ln(50)$$
$$2x = \ln 50 - \ln 4$$
$$x = \frac{1}{2}(\ln 50 - \ln 4).$$

Ex Solve $\frac{1}{[\ln(x+3)]^3} = 8$

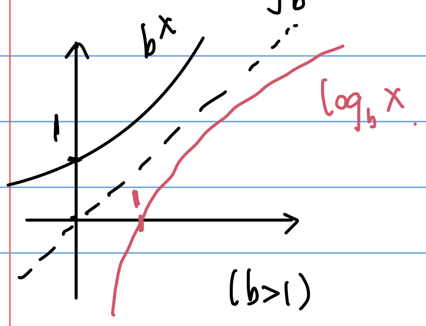
Sol.^{n}. $(\ln(x+3))^3 = (\frac{1}{2})^3$

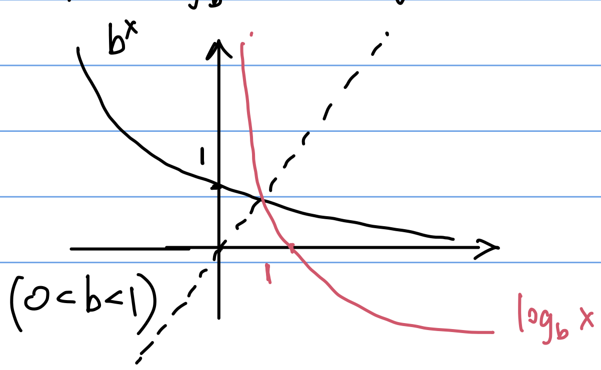
$$\ln(x+3) = \frac{1}{2}$$

• Exponentiate: $e^{\ln(x+3)} = e^{\frac{1}{2}}$.

$$x+3 = \frac{1}{2},$$
$$x = -\frac{5}{2}.$$

• Graph of $\log_b x$: $\log_b x$ is the inverse function of b^x.

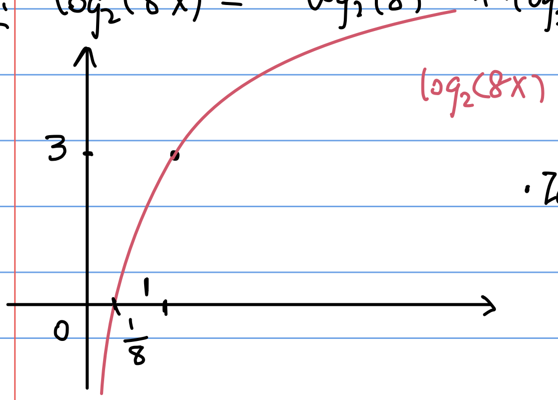




Ex Plot $\log_2(8x)$.

Sol. $\log_2(8x) = \log_2(8) + \log_2(x) = 3 + \log_2(x)$.

(Shift $\log_2(x)$ up by 3).



• Zero: $\log_2(8x) = 0$, if

$2^{\log_2(8x)} = 2^0$,

$8x = 1$, $x = \frac{1}{8}$

Ex Plot $\log_3(x^3)$.

Sol. $\log_3(x^3) = 3\log_3(x)$