Math 247 Partial Differential Equations
Final Exam

Name: Ro

- **Time**: May 6, 2:00–4:30PM. (2 hours for the exam, and 30 minutes for checking)
- Three sheets of notes allowed; no books, calculators or phones allowed.
- The exam consists of six problems. Even if you get stuck, make sure to take time for **all questions**. Partial credits are awarded for unfinished but correct work.
- Answer the questions in the spaces provided on the question sheets.
- Remember to **explain** and **justify** all your solutions, and **check** them carefully.
- Good luck!

1. (6 parts) Decide whether the following statements are true or false. You must justify your answers for full credits.

   (a) F On $\mathbb{R}^2 = \{(x,y)\}$, $(\partial_x^2 - 4\partial_x\partial_y + \partial_y)u = 0$ is a second order parabolic equation.

   coeff: $\begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix}$, $\det = -4 < 0$.

   Hyperbolic.

   (b) T Consider the following initial value problem for the wave equation on $\mathbb{R}_x$:

   $$\begin{cases} (\partial_t^2 - \partial_x^2)u(t,x) = 0, & t > 0, x \in \mathbb{R}, \\ \partial_t u(0,x) = 0, & x \in \mathbb{R}. \end{cases}$$

   Claim: it has more than one solution.

   both $u = 0$ and $u = 1$ are solutions.

   (c) T Consider $\bar{D} = \{(x,y) : x^2 + y^2 \leq 1\}$. Let $u(x,y), v(x,y)$ be two harmonic functions in $\bar{D}$ and be continuous up to the boundary.

   Claim: if $u(x,y) = v(x,y) + x$ on ∂D, then $u(x,y) = v(x,y) + x$ everywhere on $\bar{D}$.

   Let $w = u(x,y) - v(x,y) - x$.

   $\begin{cases} \Delta w = 0 \\ w|_{\partial D} = 0 \end{cases}$, max principle $\Rightarrow w \equiv 0$ on D.

   So $u(x,y) = v(x,y) + x$.

(d) F The Fourier sine series of $f(x) = x^3$ converges to f uniformly on $[0, \pi]$.

$f(\pi) = \pi^3 \neq 0$.

Boundary condition does not match.

(e) F Let D be a three-dimensional bounded domain with smooth boundary. Consider the homogeneous Neumann Poisson problem:

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ \partial_\nu u = 0, & \text{on } \partial D, \end{cases}$$

where ν is the unit outward-pointing normal.

Claim: this problem is well-posed.

$u \equiv 0$, $u \equiv 1$ are solutions,

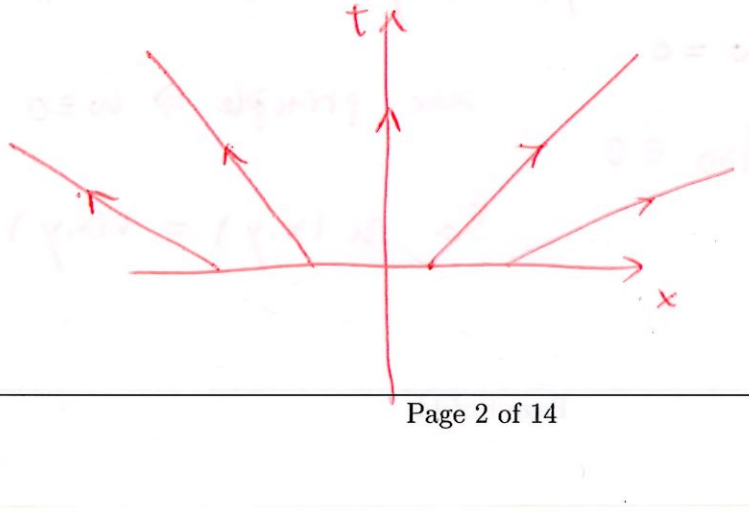
$\Rightarrow$ no uniqueness.

(f) T Consider the initial value problem for Burger's equation on $\mathbb{R}_x$:

$$\begin{cases} \partial_t u(t,x) + u(t,x)\partial_x u(t,x) = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = x, & x \in \mathbb{R}. \end{cases}$$

Claim: the characteristic lines never intersect each other for $t > 0$.

$\partial_x(u(0,x)) = 1 > 0 \Rightarrow$ expansive wave.



2. (2 parts) This problem is about the wave-type equations.

   (a) Consider the initial value problem for the wave equation:

$$\begin{cases} (\partial_t^2 - \partial_x^2)u(t,x) = 0, \quad t > 0,\ x \in \mathbb{R} \\ u(0,x) = \sin(x),\ \partial_t u(0,x) = x, \quad x \in \mathbb{R}. \end{cases}$$

Solve the problem. Unsimplified solutions yield partial credits.

D'Alembert:

$$u(t,x) = \frac{1}{2}\sin(x+t) + \frac{1}{2}(\sin(x-t) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{x}\, d\tilde{x}$$

$$= \frac{1}{2}\sin(x+t) + \frac{1}{2}\sin(x-t) + xt.$$

(b) Let D be a two-dimensional bounded domain with smooth boundary. Let the damping function $a(x,y) \geq 0$. Consider the damped wave equation on D:

$$\begin{cases} (\partial_t^2 + a(x,y)\partial_t - \Delta_{x,y})u(t,x,y) = 0, & t > 0,\ (x,y) \in D \\ u(t,x,y) = 0, & t > 0, (x,y) \in \partial D. \end{cases}$$

Here $\Delta_{x,y} = \partial_x^2 + \partial_y^2$. Let u be a solution. We define the energy of u by

$$E(t) = \frac{1}{2}\int_D (\partial_t u)^2 + (\partial_x u)^2 + (\partial_y u)^2 \, dxdy.$$

Show that $E(t) \leq E(0)$ for any $t \geq 0$.

$$\partial_t E(t) = \frac{1}{2}\int_D (\partial_t u)(\partial_t^2 u) + \underbrace{\int_D (\partial_x \partial_t u)(\partial_x u) + (\partial_y \partial_t u)(\partial_y u)}_{\int_D \partial_t \nabla u \cdot \nabla u .}$$

$$\underbrace{\int_D \partial_t \nabla u \cdot \nabla u}_{} \overset{\text{Green}}{=} -\int_D (\partial_t u)(\Delta u) + \cancel{\int_{\partial D} \overset{\overset{0}{\shortparallel}}{\overbrace{(\partial_t u)}}(\partial_\nu u}$$

$$= \frac{1}{2}\int_D (\partial_t u)(\partial_t^2 u - \Delta u)$$

$$= \frac{1}{2}\int_D - \underbrace{a(x,y)}_{\geq 0 .} \underbrace{(\partial_t u)^2}_{\geq 0 .} \, dxdy \leq 0 .$$

$$\Rightarrow E(t) \leq E(0) \ \forall t \geq 0 .$$

3. (2 parts) In $t > 0$, $x \in \mathbb{R}$, consider the heat initial value problem:

$$\partial_t u(t,x) = \partial_x^2 u(t,x).$$

with $u(0,x) = \phi(x)$.

(a) Solve the initial value problem with the specific initial data ϕ:

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x), & t > 0, x \in \mathbb{R}, \\ u(0,x) = e^{-x^2}, & x \in \mathbb{R}. \end{cases}$$

Unsimplified solutions yield partial credits.

(Hint: you may use the following formula if you really need.)

$$\int_{-\infty}^{\infty} e^{-a(y-b)^2}\,dy = \sqrt{\frac{\pi}{a}}, \quad \text{for any } a > 0, b \in \mathbb{R}.$$

One can just do

$u(t,x) = K(t,x) *_x e^{-x^2}$. $\quad K(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$.

Or more easily see

$K(\frac{1}{4}, x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$.

$\Rightarrow \sqrt{\pi}\, K(\frac{1}{4}, x) = e^{-x^2}$.

So $u(0,x)$ is just $\sqrt{\pi}\, K(t,x)$ at $t = \frac{1}{4}$,

then $u(t,x) = \sqrt{\pi}\, K(t+\frac{1}{4}, x)$.

$$\boxed{u(t,x) = \frac{1}{\sqrt{\pi(4t+1)}} e^{-\frac{x^2}{4t+1}}}$$

(b) Consider the total heat of a solution u with $\partial_x u \to 0$ as $x \to \pm\infty$:

$$H(u,t) = \int_{\mathbb{R}} u(t,x)\, dx.$$

Let ϕ be that $\int_{\mathbb{R}} \phi(x)\, dx$ is finite. Show that the total heat is conserved in time, that is, $H(u,t) = H(u,0) = \int_{\mathbb{R}} \phi(x)\, dx$ for all $t > 0$.

$$\partial_t H(u,t) = \int \partial_t u \, dx$$

$$= \int (\partial_x^2 u)\, 1 \, dx$$

$$= - \int (\partial_x u)\, \cancelto{0}{(\partial_x 1)}\, dx + \cancelto{0}{(\partial_x u)}\, 1 \Big]_{x=-\infty}^{\infty}$$

$$= 0 .$$

$$\Rightarrow H(u,t) = \text{constant}.$$

4. (2 parts) Consider the transport equation

$$t\partial_t u(t,x) + x\partial_x u(t,x) + u(t,x) = 0$$

for $t > 1$, $x \in \mathbb{R}$:

(a) Find the equation for the characteristics, and sketch some characteristics in the $\{(t,x) : t > 1, x \in \mathbb{R}\}$.

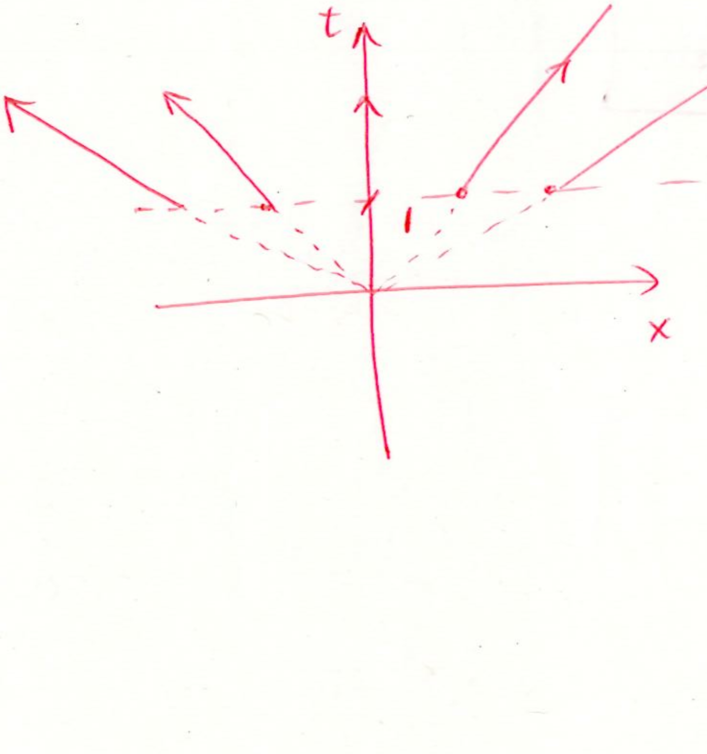
$$\gamma(s) = (t(s), x(s)) ;$$

$$\gamma'(s) = (t', x') = (t, x) .$$

$$\Rightarrow \begin{cases} t' = t \\ x' = x \end{cases} \Rightarrow \begin{cases} t = t_0 e^s \\ x = x_0 e^s , \end{cases} \xrightarrow{\text{let } t_0 = 1} \begin{cases} t = e^s , \; s = \log t . \\ x = x_0 e^s . \end{cases}$$

$$x = x_0 e^s = x_0 t .$$

$$\Rightarrow \gamma(s) = (e^s, x_0 e^s) = \boxed{\{ x = x_0 t \} .}$$



(b) Find the solution to the initial-value problem

$$\begin{cases} t\partial_t u(t,x) + x\partial_x u(t,x) + u(t,x) = 0, & t > 1, x \in \mathbb{R}, \\ u(1,x) = x, & x \in \mathbb{R}. \end{cases}$$

Note that the initial condition is imposed on $\{t = 1\}$.

$$\gamma(s) = (e^s, x_0 e^s) \qquad \gamma(0) = (1, x_0).$$

$$u_\gamma(s) = u(\gamma(s)) = u(e^s, x_0 e^s),$$

$$\partial_s u_\gamma(s) = \big((t\partial_t + x\partial_x)u\big)(\gamma(s)) = -u_\gamma(s).$$

$$\Rightarrow \begin{array}{ccccccc} u_\gamma(s) & = & u_\gamma(0)e^{-s} & = & u(1,x_0)e^{-s} & = & x_0 e^{-s} \; [illegible] \\ \shortparallel & & & & & & \shortparallel \\ u(e^s, x_0 e^s) & = & u(t,x), & & & & \frac{x}{t^2}. \; \begin{cases} x = x_0 e^s, \\ t = e^s \end{cases} \end{array}$$

$$\Rightarrow \boxed{u(t,x) = [illegible].\ \frac{x}{t^2}}$$

5. (2 parts) Let mass $m > 0$. Consider the Klein–Gordon equation on $t > 0, x \in \mathbb{R}$:

$$\begin{cases} \partial_t^2 u(t,x) - \partial_x^2 u(t,x) + m^2 u(t,x) = 0, & t > 0, x \in \mathbb{R} \\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x), & x \in \mathbb{R}. \end{cases}$$

Such PDE models the propagation of massive waves, like Higgs bosons.

(a) Let $\hat{u}(t,\xi)$ be the Fourier transform of u with respect to the x coordinate. Use Fourier transforms to find an ODE for $\hat{u}$, and find the general solution to this ODE.

FT: $\begin{cases} \partial_t^2 \hat{u}(t,\xi) + (\xi^2 + m^2)\hat{u}(t,\xi) = 0 \\ \hat{u}(0,\xi) = \hat{\phi}(\xi), \quad \partial_t \hat{u}(0,\xi) = \hat{\psi}(\xi). \end{cases}$

$\Rightarrow \hat{u}(t,\xi) = A(\xi)\sin(\sqrt{\xi^2+m^2}\,t) + B(\xi)\cos(\sqrt{\xi^2+m^2}\,t)$.

(b) Find the general solution to the Klein–Gordon equation in integral form using $\phi(x)$ and $\psi(x)$. You do not need to simplify this integral.

$$\begin{cases} \hat{u}(t,\xi) = A(\xi)\sin\left(\sqrt{\xi^2+m^2}\,t\right) + B(\xi)\cos\left(\sqrt{\xi^2+m^2}\,t\right). \\ \hat{u}(0,\xi) = \hat{\phi}(\xi), \quad \partial_t\hat{u}(0,\xi) = \hat{\psi}(\xi). \end{cases}$$

$$\hat{u}(0,\xi) = \cancel{A(\xi)}\ B(\xi) = \hat{\phi}(\xi).$$

$$\partial_t\hat{u}(t,\xi) = \sqrt{\xi^2+m^2}\,A(\xi)\cos\left(\sqrt{\xi^2+m^2}\,t\right) - \cancel{B(\xi)}\sqrt{\xi^2+m^2}\,B(\xi)\cos\left(\sqrt{\xi^2+m^2}\,t\right).$$

$$\partial_t\hat{u}(0,\xi) = \sqrt{\xi^2+m^2}\,A(\xi) = \hat{\psi}(\xi). \quad A(\xi) = \frac{\hat{\psi}(\xi)}{\sqrt{\xi^2+m^2}}.$$

$$\hat{u}(t,\xi) = \hat{\phi}(\xi)\cos\left(\sqrt{\xi^2+m^2}\,t\right) + \frac{\hat{\psi}(\xi)}{\sqrt{\xi^2+m^2}}\sin\left(\sqrt{\xi^2+m^2}\,t\right)$$

$$\boxed{u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\left[\hat{\phi}(\xi)\cos\left(\sqrt{\xi^2+m^2}\,t\right) + \frac{\hat{\psi}(\xi)}{\sqrt{\xi^2+m^2}}\sin\left(\sqrt{\xi^2+m^2}\,t\right)\right]d\xi}$$

6. (4 parts) Consider the biharmonic heat equation on a string of length 2π, $I = [-\pi, \pi]_x$:

$$\begin{cases} \partial_t u(t,x) + \partial_x^4 u(t,x) = 0, & t > 0, x \in I, \\ u(t, \pm\pi) = 0, \quad \partial_x^2 u(t, \pm\pi) = 0, & t > 0. \end{cases}$$

Here the second line is called the Navier boundary condition. Such PDE models how a metallic string straightens itself driven by curvature.

(a) Use the separation of variables with $u(t,x) = T(t)X(x)$, derive two ODE for $T(t)$ and $X(x)$. Specify the boundary conditions that $X(x)$ has.

$$T'X + TX'''' = 0.$$

$$\Rightarrow \quad -\frac{T'}{T} = \frac{X''''}{X} = \lambda \geq 0.$$

$$\begin{cases} T'(t) = -\lambda T(t) \\ X''''(x) = \lambda X(x). \end{cases}$$

$$\text{BC}: \begin{cases} X(\pm\pi) = 0, \\ X''(\pm\pi) = 0. \end{cases}$$

(∂_x^4 is formally nonnegative-definite).

$$\int (\partial_x^4 u) u \, dx = \int (\partial_x^2 u)^2 dx \geq 0.$$

(b) Solve the ODE for $T(t)$, $X(x)$, and thus find the general solution to $u(t, x)$ represented as a Fourier series.
(Hint: fourth order ODE considered here has four linearly independent solutions, but two of them do not qualify.)

- $T'(t) = -\lambda T(t) \Rightarrow T(t) = e^{-\lambda t} T(0)$.
- $X''''(x) = \lambda X(x)$. Four possiblitires:

① $X(x) = \sin(\lambda^{\frac{1}{4}} x)$.

BC: $\sin(\lambda^{\frac{1}{4}} \pi) = 0 \Leftrightarrow \lambda^{\frac{1}{4}} \in \mathbb{Z} \Leftrightarrow$ [illegible].

$\lambda = k^4,\ k \in \mathbb{Z}$.

② $X(x) = \cos(\lambda^{\frac{1}{4}} x)$.

BC. $\cos(\lambda^{\frac{1}{4}} \pi) = 0 \Leftrightarrow \lambda^{\frac{1}{4}} \in \mathbb{Z} + \frac{1}{2}$,

$\Leftrightarrow \lambda = (k + \frac{1}{2})^4,\ k \in \mathbb{Z}$.

③ $X(x) = \sinh(\lambda^{\frac{1}{4}} x) \neq 0$ for all x. Contradict BC.

④ $X(x) = \cosh(\lambda^{\frac{1}{4}} x) \neq 0$ for all x. Contradict BC.

~~$\Rightarrow T(t) X(x) = A_k \sin(kx) + B_k \cos((k+\frac{1}{2})x)$~~

$T(t) X(x) = A_k e^{-k^4 t} \sin(kx)$ or $B_k e^{-(k+\frac{1}{2})^4 t} \cos(~~[illegible]~~ (k+\frac{1}{2})t)$

$$\boxed{u(t,x) = \sum_{k=0}^{\infty} A_k e^{-k^4 t} \sin(kx) + B_k e^{-(k+\frac{1}{2})^4 t} \cos((k+\tfrac{1}{2})t).}$$

(c) Identify the principal eigenvalue of this string, that is, the smallest positive eigenvalue of ∂_x^4 on I with the Navier boundary condition. (It corresponds to the slowest way that the string straightens up.)

smallest : $k = 0$, $\lambda = (k + \frac{1}{2})^4 = (\frac{1}{2})^4 \boxed{= \frac{1}{16}.}$

(d) Solve the initial value problem

$$\begin{cases} \partial_t u(t,x) + \partial_x^4 u(t,x) = 0, & t > 0, x \in I, \\ u(t, \pm\pi) = 0, \quad \partial_x^2 u(t, \pm\pi) = 0, & t > 0, \\ u(0,x) = \sin(x), \quad x \in I. \end{cases}$$

$$u(t,x) = \sum_{k=0}^{\infty} A_k e^{-k^4 t} \sin(kx) + B_k e^{-(k+\frac{1}{2})^4 t} \cos((k+\tfrac{1}{2})x).$$

$$u(0,x) = \sum_{k=0}^{\infty} A_k \sin(kx) + B_k \cos((k+\tfrac{1}{2})x).$$

$$= \sin(x).$$

$$\Rightarrow A_k = \begin{cases} 1, & \text{if } k=1 \\ 0, & \text{if } k \neq 1 \end{cases}$$

$$B_k = 0.$$

$$\Rightarrow \boxed{u(t,x) = e^{-t} \sin(x)}.$$